Homework III
Due Date: 11/04/2024

Exercise 1. Consider the following problems.
(i) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = \cos x, \quad t > 0, x \in \mathbb{R}, \\ u(0,x) = \sin x, \partial_t u(0,x) = 1 + x, \quad x \in \mathbb{R}. \end{cases}$$

(ii) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \Delta u = t \sin y, \quad t > 0, (x,y) \in \mathbb{R}^2, \\ u(0,x,y) = x^2, \partial_t u(0,x,y) = \sin y, \quad (x,y) \in \mathbb{R}^2. \end{cases}$$

(iii) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \Delta u = 2xyz, \quad t > 0, (x,y,z) \in \mathbb{R}^3, \\ u(0,x,y,z) = x^2 + y^2 - 2z^3, \partial_t u(0,x,y,z) = 1 \quad (x,y,z) \in \mathbb{R}^3. \end{cases}$$

Exercise 2. Consider the following problems.
(i) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \quad t > 0, 0 < x < l, \\ u(t,0) = 0, u(t,l) = 0 \quad t > 0, \\ u(0,x) = \sin \frac{3\pi x}{l}, \partial_t u(0,x) = x(l-x), \quad 0 < x < l. \end{cases}$$

(ii) (1 point) Solve

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \quad t > 0, 0 < x < l, \\ u(t,0) = 0, \partial_x u(t,l) = 0, \quad t > 0. \\ u(0,x) = \frac{h}{l} x, \partial_t u(0,x) = 0, \quad 0 < x < l. \end{cases}$$

Exercise 3. Let u be a function such that

$$\partial_t^2 u - \partial_x^2 u \leq 0,$$

on the characteristic (triangular) domain

$$T := \{(t,x) : x > t, x + t < 1, t > 0\}.$$

(i) (1 point) Prove that if $\partial_t u(0,x) \leq 0$ for $0 \leq x \leq 1$, then the following maximum principle holds,

$$\max_{\overline{T}} u = \max_{0 \leq x \leq 1} u(0,x).$$

(ii) (1 point) Solve the following initial boundary value problem,

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u &= 0, \quad t > 0, 0 < x < 1, \\ u(t,0) &= 0, u(t,1) = 0, \quad t > 0, \\ u(0,x) &= 0, \partial_t u(0,x) = \sin(\pi x), \quad t > 0. \end{aligned}$$

Show that the maximum principle does not hold for the problem.

Exercise 4. Consider the semilinear wave equation

$$\partial_t^2 u - \Delta u + f(u) = 0, \quad t > 0, x \in \mathbb{R}^n.$$

(i) (1 point) Define

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t u|^2 + |\nabla u|^2 + F(u) dx,$$

where $F(u) = \int_0^u f(v)dv$. Show that $E(t)$ is constant for all $t \geq 0$.
(ii) (1 point) Define

$$e(t) := \int_{B(x_0, t_0 - t)} \frac{1}{2}(|\partial_t u|^2 + |\nabla u|^2) + F(u)dx.$$

Show that

$$\frac{de(t)}{dt} = -\int_{\partial B(x_0, t_0 - t)} \frac{1}{2}|n\partial_t u - \nabla u|^2 + F(u)dS_{(t,x)},$$

where $n = \frac{x - x_0}{|x - x_0|}$.
(iii) (1 point) If $F \geq 0$ and $u(0, x) = \partial_t u(0, x) \equiv 0$ for all $x \in B(x_0, t_0)$, then prove that $u \equiv 0$ within $\{(t, x) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$.